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On Dynamic Output Feedback Guaranteed Cost Control of Uncertain Discrete-Delay Systems: LMI Optimization Approach[1]

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Abstract. In this paper, we consider a design problem of dynamic output feedback controller for guaranteed cost stabilization of discrete-delay systems with norm-bounded time-varying parameter uncertainties. A linear-quadratic cost function is considered as a performance measure for the closed-loop system. Based on the Lyapunov second method, several stability criteria for the existence of the controller are derived in terms of linear matrix inequalities (LMIs). The solutions of the LMIs can be obtained easily using existing efficient convex optimization techniques. A numerical example is given to illustrate the proposed method.

Key Words. Discrete-delay systems, dynamic controllers, guaranteed cost stabilization, Lyapunov method, linear matrix inequalities.

1. Introduction

During the last three decades, the problems of the robust stability and performance for uncertain dynamic systems have received considerable attention; see e.g. Refs. 1–3 and references therein. One design approach to deal with uncertain dynamic systems is the guaranteed cost control, first introduced by Chang and Peng (Ref. 4). This approach has the advantage of providing an upper bound on a given performance index and thus the system

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performance degradation incurred because of the uncertainties is guaranteed to be less than this bound; see Refs. 4–7. Another front of control systems research is on time-delay systems. Delays occur often in the transmission of material or information between different parts of a system and are frequently a source of instability and poor performance. Communication systems, transmission systems, chemical procession systems, metallurgical processing systems, environmental systems, and power systems are examples of time-delay systems. Considerable efforts have been applied extensively to different aspects of time-delay systems during recent years; see the guided tours in Refs. 8–13. More recently, some significant results on guaranteed cost stabilization of time-delay systems have been proposed; see e.g. Refs. 14–18. In particular, Esfahani and Petersen (Ref. 18) introduced a design method for a class of dynamic output controllers using the LMI optimization approach. All this work has been developed for continuous time-delay systems or nondelayed discrete-time systems. Less attention has been paid to discrete-delay system. The asymptotic stability analysis of systems has been introduced in Refs. 19–22 using the characteristic equation or the Lyapunov method.

This paper is concerned with the problem of the robust guaranteed cost stabilization of discrete-delay systems with time-varying parametric uncertainties using dynamic output feedback controllers. We provide an LMI optimization problem for the existence of the controller, which renders the robust stability of the closed-loop system and guarantees an adequate level of performance. Since the proposed optimization problem ensures that a global optimum is reachable when it exists, the solutions and the upper bound of the guaranteed cost can be obtained at the same time. Utilizing the solutions, we can find easily a stabilizing dynamic output feedback controller by solving another LMI according to the procedure developed in Ref. 23.

The paper is organized as follows. In Section 2, the problem statement and notation of the guaranteed cost stabilization for discrete-delay system is introduced. Three main results and a numerical example are presented in Section 3. Finally, Section 4 concludes the paper.

Notations. The notations used in this paper are fairly standard. The superscript T denotes matrix or vector transpose. $\mathscr{R}^n$ denotes the n-dimensional Euclidean space, $\mathscr{R}^{n\times m}$ is the set of all $n\times m$ real matrices, I is the identity matrix with appropriate dimensions, and an asterisk represents the elements below the main diagonal of a symmetric block matrix. The notation $X>0$ $[X<0]$ for $X\in\mathscr{R}^{n\times n}$, means that the matrix X is symmetric and positive definite [negative definite].

2. Problem Formulation

Consider the following discrete-delay systems with time-varying uncertainties:

$$x(k+1) = (A + \Delta A(k))x(k) + (A_d + \Delta A_d(k))x(k-h) + (B + \Delta B(k))u(k), \tag{1a}$$

$$y(k) = Cx(k), \tag{1b}$$

$$x(k) = \phi(k), \qquad k \in [-h, 0], \tag{1c}$$

where $x(k) \in \mathscr{R}^n$ is the state, $u(k) \in \mathscr{R}^m$ is the control, $y(k) \in \mathscr{R}^l$ is the output, h is delay time in the system, A, A_d, B, C are constant matrices with appropriate dimensions, $\Delta A(k), \Delta A_d(k), \Delta B(k)$ are real-valued matrices representing the time-varying parameter uncertainties in the system, and $\phi(k)$ is a vector-valued initial condition function.

Assume that the triplet (A, B, C) is stabilizable and detectable and that the time-varying uncertainties are of the form

$$\Delta A(k) = D_1 F_1(k) E_1, \qquad \Delta A_d(k) = D_2 F_2(k) E_2, \tag{2a}$$

$$\Delta B(k) = D_3 F_3(k) E_3, \tag{2b}$$

where $D_1, D_2, D_3, E_1, E_2, E_3$ are known constant real matrices with appropriate dimensions and $F_1(k), F_2(k), F_3(k)$ are unknown matrix functions which are bounded,

$$F_1^T(k) F_1(k) \leq I, \qquad F_2^T(k) F_2(k) \leq I, \tag{3a}$$

$$F_3^T(k) F_3(k) \leq I, \qquad \forall k \geq 0. \tag{3b}$$

Associated with the system (1) is the following quadratic cost function:

$$J = \sum_{k=0}^{\infty} [x^T(k) Q_1 x(k) + u^T(k) Q_2 u(k)], \tag{4}$$

where $Q_1 \in \mathscr{R}^{n \times n}$ and $Q_2 \in \mathscr{R}^{m \times m}$ are given positive-definite matrices.

Now, in order to stabilize the system (1), let us consider the following dynamic output feedback controller:

$$\xi(k+1) = A_c \xi(k) + B_c y(k), \tag{5a}$$

$$u(k) = C_c \xi(k) + D_c y(k), \tag{5b}$$

$$\xi(0) = 0, \tag{5c}$$

where $\xi(k) \in \mathscr{R}^k$ and A_c, B_c, C_c, D_c are constant matrices with proper dimensions. Then, for all the admissible uncertainties and time delay h, the problem is to find the parameters of the dynamic controller (5) such that the resulting closed-loop system is globally asymptotically stable and the closed-loop value of the cost function (4) satisfies $J \leq J^*$, where J^* is some specified constant.

Definition 2.1. For the system (1) and cost function (4), if there exist a control law $u^*(k)$ and a positive J^* such that the resulting closed-loop system is asymptotically stable and the closed-loop value of the cost function (4) satisfies $J \leq J^*$, then J^* is said to be a guaranteed cost and $u^*(k)$ is said to be guaranteed cost control law of the system (1) and cost function (4).

Before proceeding further, we will state some well-known lemmas.

Lemma 2.1. Schur complements. See Ref. 26. Given the constant symmetric matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then $\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0.$$

Lemma 2.2. See Ref. 23. Consider the problem of finding some matrix K of compatible dimensions such that

$$\Psi + \Pi K^T \Theta^T + \Theta K \Pi^T < 0, \tag{6}$$

where Ψ is any symmetric matrix and Π and Θ are matrices with appropriate dimensions. Let $\bar{\Pi}$ and $\bar{\Theta}$ be the matrices whose columns are formed by the bases of the null spaces of Π and Θ. Then, the above inequality (6) is solvable for K if and only if

$$\bar{\Pi}^T \Psi \bar{\Pi} < 0, \qquad \bar{\Theta}^T \Psi \bar{\Theta} < 0. \tag{7}$$

3. Main Results

In this section, we establish several criteria for the guaranteed cost stabilization of the system (1) with dynamic output feedback controller (5) using the Lyapunov method and the LMI technique.

Let us define the augmented state vector and the controller gain matrix $K \in \mathscr{R}^{(m+k)\times(m+k)}$ as

$$x_c(k) = \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}, \qquad K = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \tag{8}$$

The closed-loop system (1) with the controller (5) can be described in the form

$$x_c(k+1) = \hat{A} x_c(k) + \hat{A}_d x(k-h). \tag{9}$$

Here,

$$\hat{A} = \bar{A} + \bar{B}K\bar{C} + \bar{D}_1 F_1(k)\bar{E}_1 + \bar{D}_3 F_3(k)\bar{E}_3 K\bar{C}, \tag{10a}$$

$$\hat{A}_d = \bar{A}_d + \bar{D}_2 F_2(k) E_2, \tag{10b}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \tag{11a}$$

$$\bar{D}_1 = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \qquad \bar{D}_2 = \begin{bmatrix} D_2 \\ 0 \end{bmatrix}, \qquad \bar{D}_3 = \begin{bmatrix} D_3 \\ 0 \end{bmatrix}, \tag{11b}$$

$$\bar{A}_d = \begin{bmatrix} A_d \\ 0 \end{bmatrix}, \qquad \bar{E}_1 = [E_1 \quad 0], \qquad \bar{E}_3 [E_3 \quad 0]. \tag{11c}$$

The corresponding closed-loop cost function is

$$\begin{aligned} J &= \sum_{k=0}^{\infty} x_c^T(k) \begin{bmatrix} Q_1 + C^T D_c^T Q_2 D_c C & C^T D_c^T Q_2 C_c \\ * & C_c^T Q_2 C_c \end{bmatrix} x_c(k) \\ &= \sum_{k=0}^{\infty} x_c^T(k) \left(\begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} + \bar{C}^T K^T \begin{bmatrix} Q_2 & 0 \\ 0 & 0 \end{bmatrix} K\bar{C} \right) x_c(k) \\ &\equiv \sum_{k=0}^{\infty} x_c^T(k) Q x_c(k). \end{aligned} \tag{12}$$

Then, we have the following theorem.

Theorem 3.1. For given $S > 0$, $Q_1 > 0$, $Q_2 > 0$, the dynamic controller $u(k)$ given by (5) is a guaranteed cost control law for the system (1) if there exist a matrix $P > 0$ such that the following LMI holds:

$$W_0 = \begin{bmatrix} \hat{A}^T P \hat{A} - P + R + Q & \hat{A}^T P \hat{A}_d \\ \hat{A}_d^T P \hat{A} & \hat{A}_d^T P \hat{A}_d - (S + E_2^T E_2) \end{bmatrix} < 0, \tag{13}$$

where the matrix R is defined as

$$R = \begin{bmatrix} S + E_2^T E_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, the upper bound of the cost function (4) is as follows:

$$J \le x_c^T(0) P x_c(0) + \sum_{i=1}^{h} x_c^T(-i) R x_c(-i) \triangleq J^*. \tag{14}$$

Proof. Define a Lyapunov function of the form

$$V(x_c(k)) := x_c^T(k)Px_c(k) + \sum_{i=k-h}^{k-1} x_c^T(i)Rx_c(i). \tag{15}$$

By evaluating the corresponding Lyapunov difference along the solutions of the system (9), we get

$$\begin{aligned} \Delta V_k &= V_{k+1} - V_k \\ &= z^T(k)\begin{bmatrix} \hat{A}^T P\hat{A} - P + R & \hat{A}^T P\hat{A}_d \\ \hat{A}_d^T P\hat{A} & \hat{A}_d^T P\hat{A}_d - (S + E_2^T E_2) \end{bmatrix} z(k) \\ &= z^T(k)W_0 z(k) - x_c^T(k)Qx_c(k), \end{aligned} \tag{16}$$

where

$$z(k) = \begin{bmatrix} x_c(k) \\ x(k-h) \end{bmatrix}.$$

Noting that $Q \geq 0$, the Lyapunov difference is negative if there exists a positive-definite matrix P such that W_0 is a negative definite matrix. This implies that there exist a positive scalar γ such that

$$\Delta V_k < -\gamma \|x(k)\|^2,$$

which guarantees the asymptotic stability of the system by Lyapunov stability theory. Furthermore, from (16), we have

$$x_c^T(k)Qx_c(k) \leq -\Delta V_k = V(k) - V(k+1).$$

Summing both sides of the above inequality from 0 to ∞ leads to

$$\sum_{k=0}^{\infty} x_c^T(k)Qx_c(k) \leq V(0) - V(\infty).$$

Since the asymptotic stability of the system has been established already, we conclude that

$$V(k) \to 0, \qquad \text{as } k \to \infty.$$

Hence, we have

$$J \leq V(0) = J^*. \tag{17}$$

This completes the proof. □

Since the LMI (13) contains uncertain matrices, it is difficult to test whether the inequality is satisfied. In the following, we give a verifiable equivalent characterization of the condition in Theorem 3.1.

By Lemma 2.1 (Schur complements), the fact is that the inequality $W_0 < 0$ is equivalent to the following matrix inequality:

$$W_1 \equiv \begin{bmatrix} -P^{-1} & \hat{A} & \hat{A}_d \\ * & -P+R+Q & 0 \\ * & * & -(S+E_2^T E_2) \end{bmatrix}$$

$$= \begin{bmatrix} -P^{-1} & \begin{pmatrix} \bar{A}+\bar{B}K\bar{C}+\bar{D}_1F_1\bar{E}_1 \\ +\bar{D}_3F_3\bar{E}_3K\bar{C} \end{pmatrix} & \bar{A}_d+\bar{D}_2F_2E_2 \\ * & -P+R+Q & 0 \\ * & 0 & -(S+E_2^T E_2) \end{bmatrix} < 0. \tag{18}$$

Using the known fact that

$$U\Delta V^T + V\Delta U^T \leq \epsilon UU^T + \epsilon^{-1} VV^T, \qquad \epsilon > 0,$$

for any matrices U, V, Δ with $\Delta^T\Delta \leq I$, we can eliminate the unknown factor $F_i(k)$ due to the parameter uncertainties. Then, we have

$$W_1 \leq W_2 = \begin{bmatrix} \begin{pmatrix} -P^{-1}+\bar{D}_1\bar{D}_1^T \\ +\bar{D}_2\bar{D}_2^T+\bar{D}_3\bar{D}_3^T \end{pmatrix} & \bar{A}+\bar{B}K\bar{C} & \bar{A}_d \\ * & \begin{pmatrix} -P+R+Q+\bar{E}_1^T\bar{E}_1 \\ +\bar{C}^TK^T\bar{E}_3^T\bar{E}_3K\bar{C} \end{pmatrix} & 0 \\ * & * & -S \end{bmatrix}. \tag{19}$$

For simplicity, define $\bar{D}$ as

$$\bar{D} = \bar{D}_1\bar{D}_1^T + \bar{D}_2\bar{D}_2^T + \bar{D}_3\bar{D}_3^T.$$

Here, we decompose the term $R+Q+\bar{E}_1^T\bar{E}_1+\bar{C}^TK^T\bar{E}_3^T\bar{E}_3K\bar{C}$ of the $(2,2)$ entry in the matrix W_2 as

$$\begin{aligned} &R+Q+\bar{E}_1^T\bar{E}_1+\bar{C}^TK^T\bar{E}_3^T\bar{E}_3K\bar{C} \\ &= \begin{bmatrix} S+E_1^TE_1+E_2^TE_2+Q_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_c^T(E_3^TE_3+Q_2)C_c \end{bmatrix} \\ &\equiv \bar{S}+\bar{C}^TK^T\bar{Q}K\bar{C}, \end{aligned} \tag{20}$$

where

$$\bar{Q} = \begin{bmatrix} E_3^T E_3 + Q_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Defining

$$D_e = (D_1 D_1^T + D_2 D_2^T + D_3 D_3^T)^{1/2},$$
$$Q_e = (E_3^T E_3 + Q_2)^{1/2},$$
$$S_e = (S + E_1^T E_1 + E_2^T E_2 + Q_1)^{1/2},$$

and using Lemma 2.1 and (20), the fact that $W_2 < 0$ is equivalent to

$$\begin{bmatrix} -P^{-1} & \hat{D} & \bar{A} + \bar{B}K\bar{C} & 0 & 0 & \bar{A}_d \\ * & -I & 0 & 0 & 0 & 0 \\ * & * & -P & \hat{S} & \bar{C}^T K^T \hat{Q}^T & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -S \end{bmatrix} < 0, \tag{21}$$

where

$$\hat{D} = \begin{bmatrix} D_e & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} S_e & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q_e & 0 \\ 0 & 0 \end{bmatrix}.$$

The inequality (21) can be decomposed as follows:

$$\Psi + \Pi K \Theta^T + \Theta K^T \Pi^T < 0, \tag{22}$$

where

$$\Psi = \begin{bmatrix} -P^{-1} & \hat{D} & \bar{A} & 0 & 0 & \bar{A}_d \\ * & -I & 0 & 0 & 0 & 0 \\ * & * & -P & \hat{S} & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -S \end{bmatrix}, \quad \Pi = \begin{bmatrix} \bar{B} \\ 0 \\ 0 \\ 0 \\ \hat{Q} \\ 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 \\ 0 \\ \bar{C}^T \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{23}$$

It is clear that the inequality (22) is the criterion for the guaranteed cost stabilization of the closed-loop system (9). If we can find an appropriate $P > 0$, the inequality (22) becomes an LMI with respect to K, which can be solved easily by various efficient convex optimization algorithms (Ref. 26). To the above end, we can invoke now Lemma 2.2 to obtain solvability conditions for (22). Then, the inequality (22) is equivalent to

$$\Pi_\perp^T \Psi \Pi_\perp < 0 \tag{24}$$

and

$$\Theta_\perp^T \Psi \Theta_\perp < 0, \tag{25}$$

where $\Pi_\perp$ and $\Theta_\perp$ are any matrices whose columns form the bases of the null spaces of Π and Θ, respectively.

Now, we will find the matrix P which satisfies (24) and (25). Let us choose $[W_1^T, W_2^T]^T$ and W_3 as the orthogonal complement of $[B^T, Q_e^T]^T$ and C^T, respectively. Then,

$$\Pi_\perp = \begin{bmatrix} \begin{bmatrix} W_1 \\ 0 \end{bmatrix} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ W_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad \Theta_\perp = \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \begin{bmatrix} W_3 \\ 0 \end{bmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}. \tag{26}$$

Now, proceeding along lines similar to those in the Gahinet and Apkarian work (for details, see Ref. 23), we obtain simplified conditions equivalent to (24) and (25) utilizing the internal structure of the augmented matrices such as (11a), (23), and (26).

To simplify the condition (24), we partition P and P^{-1} as

$$P = \begin{bmatrix} Y & N \\ N^T & \bullet \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} X & M \\ M^T & \bullet \end{bmatrix}, \tag{27}$$

where $X, Y \in \mathscr{R}^{n\times n}$, $M, N \in \mathscr{R}^{n\times k}$, $\bullet$ means irrelevant, and

$$MN^T = I - XY. \tag{28}$$

Using Lemma 2.1 and carrying out block multiplication with (26) and (27), the inequality (24) simplifies to the following LMI:

$$\mathscr{W}_x^T \begin{bmatrix} -X + AXA^T & D_e & AXS_e & 0 & A_d \\ * & -I & 0 & 0 & 0 \\ * & * & -I + S_e^T X S_e & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -S \end{bmatrix} \mathscr{W}_x < 0, \tag{29}$$

where

$$\mathscr{W}_x = \begin{bmatrix} W_1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ W_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Also, performing the same procedure on (25), the inequality (25) simplifies to

$$\mathscr{W}_y^T \begin{bmatrix} -Y + A^T YA & S_e & A^T YA_d & A^T YD_e \\ * & -I & 0 & 0 \\ * & * & -S + A_d^T YA_d & A_d^T YD_e \\ * & * & * & -I + D_e^T YD_e \end{bmatrix} \mathscr{W}_y < 0, \tag{30}$$

where

$$\mathscr{W}_y = \begin{bmatrix} W_3 & 0 \\ 0 & I \end{bmatrix}.$$

Since P is a positive-definite matrix, we need an additional condition on X and Y (Refs. 23–25),

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0. \tag{31}$$

On the other hand, the upper bound J^* given in (17) of the cost function (4) can be rewritten as

$$J \leq x^T(0)\, Yx(0) + \sum_{i=1}^{h} x^T(-i)(S + E_2^T E_2)x(-i) \triangleq J^*. \tag{32}$$

Now, we can summarize our second result in following theorem.

Theorem 3.2. For the given uncertain discrete-delay system (1) with dynamic output feedback controller (5) and $S>0$, if there exist $X>0$ and $Y>0$ such that the three LMIs (29), (30), (31) hold, there exists a $P>0$ satisfying the inequalities (24) and (25). Then, the parameter K satisfying the inequality (22) also exists by Lemma 2.2. Furthermore, the controller parameter matrix K is a control law for the robust guaranteed cost stabilization of the uncertain system (9) and the corresponding closed-loop value of the cost function satisfied $J \leq J^*$, in which J^* is given by (32).

Theorem 3.2 presents a method of designing a dynamic output feedback guaranteed cost controller. The following theorem presents a method of selecting a controller minimizing the upper bound of the guaranteed cost (32).

Theorem 3.3. Consider the system (9) with cost function (4). If the following LMI optimization problem:

$$\min_{X>0,\,Y>0,\,\alpha>0} \quad \alpha, \tag{33a}$$

$$\text{s.t.} \quad \text{(i)} \quad \text{LMIs } (29), (30), (31), \tag{33b}$$

$$\text{(ii)} \quad \begin{bmatrix} -\alpha & x^T(0)\,Y \\ Yx(0) & -Y \end{bmatrix} < 0, \tag{33c}$$

has a positive solution set (X, Y, α), then the control law (5) is an optimal robust guaranteed cost control law which ensures the minimization of the guaranteed cost (32) for the system (9).

Proof. By Theorem 3.2, (i) in (33) is clear. Also, it follows from the Lemma 2.1 that (ii) in (33) are equivalent to

$$x^T(0)\,Yx(0) < \alpha.$$

Hence, it follows from (32) that

$$J^* < \alpha + \beta,$$

where

$$\beta = \sum_{i=1}^{h} x^T(-i)(S + E_2^T E_2)x(-i).$$

Thus, the minimization of α implies the minimization of the guaranteed cost for the system (9). This convex optimization problem guarantees that a global optimum, when it exists, is reachable (Ref. 26). □

Remark 3.1. To find the controller parameter matrix K, first find a solution (X, Y) of the optimization problem (33) and second find two full-column-rank matrices $M, N \in \mathcal{R}^{n\times k}$ satisfying (28). Then, we can find the unique matrix P from

$$\begin{bmatrix} Y & I \\ N^T & 0 \end{bmatrix} = P \begin{bmatrix} I & X \\ 0 & M^T \end{bmatrix}. \tag{34}$$

For the matrix P, the controller parameter matrix K can be obtained easily by solving the LMI (22). Moreover, if

$$\mathrm{rank}(I - XY) = k$$

for the solution matrices X and Y, the order of the dynamic controllers is k (Ref. 23).

Remark 3.2. In this paper, in order to solve the LMIs, we utilize the Matlab LMI Control Toolbox (Ref. 27), which implements state-of-the-art interior-point algorithms, which are significantly faster than classical convex optimization algorithms (Ref. 26).

Remark 3.3. In Ref. 18, the problem of the dynamic output controller design for guaranteed cost stabilization of a class of time-delay systems in the continuous-time domain has been studied. However, the controller designed is strictly a proper one, i.e, $D_c = 0$. This is a special case of our design approach.

Example 3.1. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1.2 \\ -1.2 & 0 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = [1 \quad 0],$$

$$\Delta A = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} F_1(k)[1 \quad 1], \qquad \Delta A_d = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} F_2(k)[1 \quad 1],$$

$$\Delta B = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} F_3(k), \qquad h = 5, \qquad S = I,$$

where

$$F_i^T(k) F_i(k) \leq I, \qquad i = 1, 2, 3;$$

the initial condition of the system is as follows:

$$x(t) = [1, \; -1]^T, \qquad -5 \leq k \leq 0.$$

Actually, when the control input is not applied to this system, it can be seen easily that the system trajectory goes to infinity as $k \to \infty$. Here, associated with this system is the cost function (4) with $Q_1 = 0.1I$ and $Q_2 = 0.1$. Also, let

$S = I$. From the relation

$$\beta = \sum_{i=1}^{h} x^T(-i)(S + E_2^T E_2)x(-i),$$

we have $\beta = 10$.

Now, solving the optimization problem of Theorem 3.3 using the Matlab LMI Control Toolbox (Ref. 27), we can get the solution of the problem as

$$X = \begin{bmatrix} 0.3180 & -0.0065 \\ -0.0065 & 0.1210 \end{bmatrix}, \quad Y = \begin{bmatrix} 4.0917 & -1.5329 \\ -1.5329 & 11.3481 \end{bmatrix}, \quad \alpha = 18.5056,$$

and a pair of solution matrices satisfying (28) is

$$M = \begin{bmatrix} -0.2087 & 0.6123 \\ 0.6123 & 0.2087 \end{bmatrix}, \qquad N = \begin{bmatrix} 0.4655 & -0.3495 \\ -0.8399 & 0.6306 \end{bmatrix}.$$

Then, the positive-definite solution P from the relation (34) can be obtained as

$$P = \begin{bmatrix} 4.0917 & -1.5329 & 0.4655 & -0.3495 \\ -1.5329 & 11.3481 & -0.8399 & 0.6306 \\ 0.4655 & -0.8399 & 0.2297 & -0.1724 \\ -0.3495 & 0.6306 & -0.1724 & 0.1294 \end{bmatrix}.$$

Therefore, by solving the LMI (22) with respect to K, we can find a stabilizing guaranteed cost dynamic output feedback controller as

$$K = \left[\begin{array}{c|cc} 1.0342 & -0.0684 & 0.0514 \\ \hline -1.5461 & -0.5642 & 0.4237 \\ 1.1309 & 0.5571 & -0.4180 \end{array}\right],$$

and the optimal guaranteed cost of the closed-loop system is

$$J^* = \alpha + \beta = 28.5056.$$

The responses of the states and control input of the above system with $F_i(k) = 1, i = 1, 2, 3$, are given Figs. 1 and 2.

4. Conclusions

In this paper, the guaranteed cost stabilization problem for uncertain discrete-delay systems has been investigated based on the Lyapunov method.

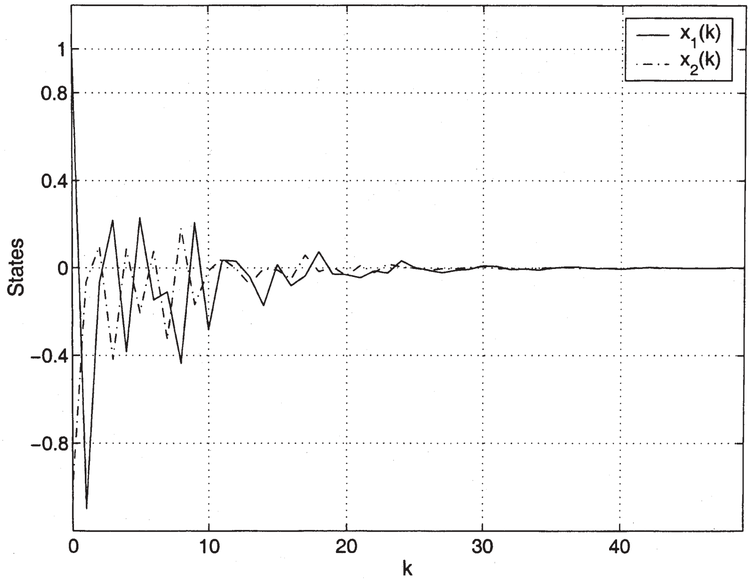

Fig. 1. State responses of the closed-loop system.

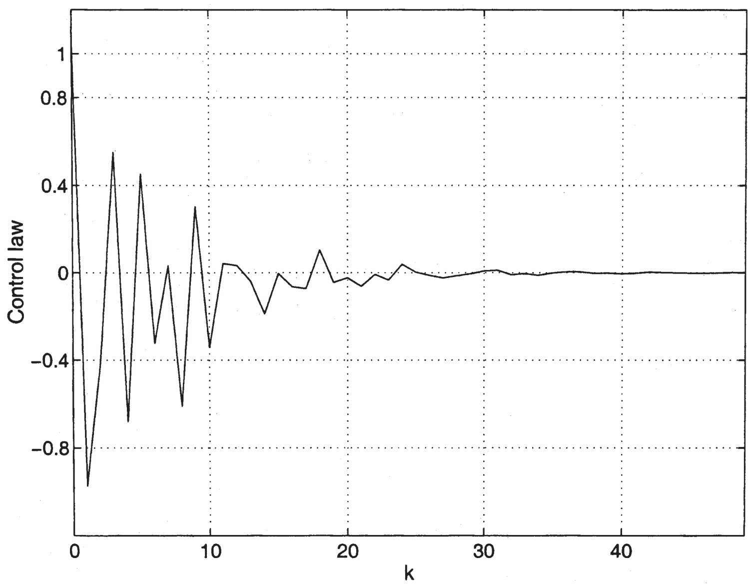

Fig. 2. Control input.

A dynamic output feedback controller for the stabilization of the system has been proposed. It is shown that selecting an optimal controller in the sense of guaranteeing the asymptotic stability of the closed-loop system and minimizing the upper bound of quadratic performance index lead to a convex optimization problem with some LMIs restrictions, which can be solved by various efficient algorithms.

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